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一类分数阶混合(p,q)-积分差分系统解的存在性和稳定性*

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摘　要：研究了一类分数阶混合(p,q)-积分差分系统解的存在性、唯一性以及有限时间稳定性．首先，利用Banach压缩映射原理和Krasnoselskii不动点定理分别得到了该系统解的唯一性和存在性的充分条件；进一步，验证了该系统解在特定条件下的有限时间稳定性；为验证理论结果的有效性，给出实例并通过其数值结果验证了主要结论的可行性与适用性．

关键词：分数阶(p,q)-积分；分数阶(p,q)-差分；初值问题；不动点定理

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Existence and stability of solutions for a class of fractional hybrid (p,q)-integral-difference systems

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Abstract: The existence, uniqueness, and finite-time stability of solutions are investigated for a class of fractional hybrid (p, q)-integral-difference systems. Firstly, sufficient conditions for both the uniqueness and existence of solutions are derived using Banach's contraction mapping principle and Krasnoselskii's fixed point theorem, respectively. Subsequently, the finite-time stability of the system solutions is rigorously verified under specified conditions. To demonstrate the theoretical results, a numerical example is presented along with computational results to validate the feasibility and practical applicability of the main theoretical findings.

Key words: fractional (p,q)-integral; fractional (p,q)-difference; initial value problem; fixed point theorem

在过去的几十年里,量子微积分的理论发展及其应用研究取得了显著进展．作为经典微积分的离散推广，q-微积分在数学物理、组合数学等领域展现出强大的应用价值．在此基础上,Chakrabarti et al.(1991)提出了更具一般性的双参数(p,q)-微积分理论，该理论通过引入两个独立的量子参数,为描述复杂系统提供了更灵活的数学工具．(p,q)-微积分理论的发展呈现出多学科交叉的特点：在基础理论层面,Njionou Sadjang(2018)系统建立了(p,q)-微积分的基本定理和(p,q)-Taylor展开公式；在方程研究方面，Kamsrisuk et al.(2018)和Promsakon et al.(2018)开创性地研究了(p,q)-差分方程的边值问题；在算子逼近领域，

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Rahman et al.（2018）和Bin Jebreen et al.（2019）则深入探讨了(p,q)-Bernstein算子的收敛性质. 此外，分数阶(p,q)-差分方程作为该理论的重要延伸，成功融合了离散系统的动态特征与连续系统的记忆效应，在复杂系统建模中展现了独特优势. 以量子信息科学为例，这类方程能够精确刻画量子计算中的关键过程：通过q-参数描述量子门操作的离散特性，同时利用p-参数捕捉环境噪声的连续记忆效应，为量子比特的非马尔可夫动力学建模提供了新的数学框架. 这种双参数结构使得(p,q)-微积分在解决工程与科学中的复杂问题时具有不可替代的理论价值和实际意义. 尽管近年来离散分数阶微积分理论已取得显著进展，但其理论体系的完整性和深度仍远不及连续分数阶微积分. 特别是在双参数量子微积分领域，分数阶(p,q)-微积分的研究尚处于起步阶段，相关理论框架和实际应用仍有待深入探索.

Agarwal et al.（2022）研究了一类涉及Caputo型q-导数和不同阶的Riemann-Liouville型q-积分的分数阶混合q-差分方程

$$\begin{cases} {}^{c}D_q^{\alpha}\left[u(x)-f(x,u(x))\right]=ag(x,u(x))+bI_q^{\delta}h(x,u(x)),\ 0<q<1,\ 0\leqslant x\leqslant 1,\\ u(0)=u_0\displaystyle\int_0^{\eta}\frac{(\eta-qs)^{(\gamma-1)}}{\Gamma_q(\gamma)}u(s)\mathrm{d}_qs,\ \ \gamma>0,\ 0<\eta<1,\ u_0\in\mathbb{R}\end{cases}$$

的解的存在性和稳定性，其中${}^{c}D_q^{\alpha}$和I_q^{δ}分别表示α阶的Caputo型q-导数和Riemann-Liouville型q-积分且$\alpha,\delta\in(0,1)$, $a,b\in\mathbb{R}$, $f,\ g,\ h$: $[0,1]\times\mathbb{R}\to\mathbb{R}$是连续函数.

Mesmouli et al.（2024）研究了一类分数阶(p,q)-积分差分系统

$$\begin{cases} {}^{c}D_{p,q}^{\lambda}(\boldsymbol{y}(t)-\boldsymbol{f}_1(t,y(t)))=\boldsymbol{A}\boldsymbol{f}_2(p^{\lambda}t,\boldsymbol{y}(p^{\lambda}t))+\boldsymbol{B}\Im_{p,q}^{\delta}\boldsymbol{f}_3(p^{\lambda+\delta}t,\boldsymbol{y}(p^{\lambda+\delta}t)),\ 0\leqslant t\leqslant 1,\\ \boldsymbol{y}(0)=\dfrac{\boldsymbol{C}}{p^{\binom{\sigma}{2}}\Gamma_{p,q}(\sigma)}\displaystyle\int_0^{\zeta}(\zeta-qs)_{p,q}^{(\sigma-1)}\boldsymbol{y}(ps)\mathrm{d}_{p,q}s,\ \sigma>0,\ 0<\zeta<1\end{cases}$$

的存在性和唯一性，其中$0<q<p\leqslant 1$，${}^{c}D_{p,q}^{\lambda}$表示λ阶Caputo型分数阶(p,q)-导数，$\Im_{p,q}^{\delta}$表示δ阶的Riemann-Liouville型(p,q)-积分，$\lambda,\delta\in(0,1)$, $\boldsymbol{A},\boldsymbol{B}$和$\boldsymbol{C}$是常数矩阵，$\boldsymbol{f}_1,\ \boldsymbol{f}_2,\ \boldsymbol{f}_3$: $[0,1]\times\mathbb{R}^n\to\mathbb{R}^n$是连续函数.

受以上工作及文献（Nagajothi et al.，2021；Houas et al.，2023；Khalid et al.，2024）启发，本文利用Banach压缩映射原理和Krasnoselskii不动点定理研究如下一类混合分数阶(p,q)-积分差分系统

$$\begin{cases} {}^{c}D_{p,q}^{\alpha}\left({}^{c}D_{p,q}^{\omega}\left({}^{c}D_{p,q}^{\theta}\left(\boldsymbol{y}(t)-\boldsymbol{f}_1\left(t,\boldsymbol{y}(t)\right)\right)\right)\right)=\boldsymbol{A}I_{p,q}^{\tau}\boldsymbol{f}_2\left(p^{(\tau+\alpha+\omega+\theta)}t,\boldsymbol{y}\left(p^{(\tau+\alpha+\omega+\theta)}t\right)\right)\\ \qquad\qquad +\boldsymbol{B}I_{p,q}^{\delta}\boldsymbol{f}_3\left(p^{(\delta+\alpha+\omega+\theta)}t,\boldsymbol{y}\left(p^{(\delta+\alpha+\omega+\theta)}t\right)\right),\\ y(0)=\dfrac{\boldsymbol{C}}{p^{\binom{\beta+\alpha+\omega+\theta}{2}}\Gamma_{p,q}\left(\beta+\alpha+\omega+\theta\right)}\displaystyle\int_0^{\xi}\left(\xi-qs\right)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\boldsymbol{y}\left(ps\right)\mathrm{d}_{p,q}s,\\ {}^{c}D_{p,q}^{\omega}\left({}^{c}D_{p,q}^{\theta}\left(\boldsymbol{y}(t)-\boldsymbol{f}_1\left(t,\boldsymbol{y}(t)\right)\right)\right)\Big|_{t=0}=0,\qquad {}^{c}D_{p,q}^{\theta}\left(\boldsymbol{y}(t)-\boldsymbol{f}_1\left(t,\boldsymbol{y}(t)\right)\right)\Big|_{t=0}=0,\end{cases}\tag{1}$$

解的存在性和稳定性，其中$0<q<p\leqslant 1$，$0<\{\alpha,\delta,\tau,\omega,\theta\}\leqslant 1$，$\beta\in(0,1]$，$\xi\in(0,1)$，${}^{c}D_{p,q}^{\varphi}$表示$\varphi$阶Caputo型分数阶$(p,q)$-导数，$I_{p,q}^{\psi}$表示$\psi$阶的Riemann-Liouville型$(p,q)$-积分，$\varphi,\psi\in(\alpha,\delta,\tau,\omega,\theta)$, $\boldsymbol{A},\boldsymbol{B}$和$\boldsymbol{C}$是常数矩阵，$\boldsymbol{f}_1,\ \boldsymbol{f}_2,\ \boldsymbol{f}_3$: $[0,1]\times\mathbb{R}^n\to\mathbb{R}^n$是连续函数，$p^{\binom{\beta+\alpha+\omega+\theta}{2}}=\dfrac{(\beta+\alpha+\omega+\theta)(\beta+\alpha+\omega+\theta-1)}{2}$.

1　预备知识

定义 1（Ferreira，2010）　设$0<q<p\leqslant 1$为常数，则(p,q)-微积分具有如下基本关系式

$$[m]_{p,q}:=\begin{cases}\dfrac{p^m-q^m}{p-q}=p^{m-1}[m]_{\frac{q}{p}}, & m\in\mathbb{N}^{+},\\ 1, & m=0,\end{cases}$$

其中

$$[m]_{\frac{q}{p}}:=\left[1-\left(\frac{q}{p}\right)^m\right]\Big/\left(1-\frac{q}{p}\right),$$

$$[m]_{p,q}! := \begin{cases} [m]_{p,q}[m-1]_{p,q}\cdots[1]_{p,q} = \prod_{i=1}^{m}\frac{p^i - q^i}{p-q}, & m \in \mathbb{N}^+, \\ 1, & m = 0. \end{cases}$$

类似地，对于幂函数$(a-b)_q^{(n)}$，$\lambda \in \mathbb{R}$的(p,q)-微积分具有如下关系式

$$(a-b)_{p,q}^{(\lambda)} := p^{\binom{\lambda}{2}}(a-b)_{\frac{q}{p}}^{(\lambda)} = a^\lambda p^{\binom{\lambda}{2}}\prod_{i=0}^{\infty}\frac{a - b\left(\frac{q}{p}\right)^i}{a - b\left(\frac{q}{p}\right)^{\lambda+i}},\quad 0<b<a,$$

其中$p^{\binom{\lambda}{2}} := \dfrac{\lambda(\lambda-1)}{2}$.

定义 2（Njionou Sadjang，2018） 设$0<q<p<1$，函数w的(p,q)-导数定义为

$$D_{p,q}w(t) := \frac{w(pt)-w(qt)}{(p-q)t},\quad t\neq 0,$$

且$(D_{p,q}w)(0) = \lim_{t\to 0}(D_{p,q}w)(t)$，使得$w$在0点可微. 此外，高阶$(p,q)$-导数$D_{p,q}^n w(t)$定义为

$$D_{p,q}^n w(t) = \begin{cases} D_{p,q}D_{p,q}^{n-1}w(t), & n\in\mathbb{N}^+, \\ w(t), & n = 0. \end{cases}$$

定义 3（Njionou Sadjang，2018） 设$0<q<p<1$且w为实数t的任意函数. 函数w的(p,q)-积分定义为

$$\mathrm{I}_{p,q}w(t) = \int_0^t w(s)\mathrm{d}_{p,q}s = (p-q)t\sum_{i=0}^{\infty}\frac{q^i}{p^{i+1}}w\left(\frac{q^i}{p^{i+1}}t\right),$$

使得右侧级数收敛，则称w在区间$[0,t]$上(p,q)-可积.

定义 4（Soontharanon et al.，2020） 对$\lambda\in\mathbb{R}$，(p,q)-伽马函数由下式给出

$$\Gamma_{p,q}(\lambda) = \frac{(p-q)_{p,q}^{(\lambda-1)}}{(p-q)^{\lambda-1}},$$

其中$\Gamma_{p,q}(\lambda+1) = [\lambda]_{p,q}\Gamma_{p,q}(\lambda)$.

定义 5（Soontharanon et al.，2020） 设$\lambda>0$，$0<q<p\leqslant 1$，且w: $[0,1]\to\mathbb{R}$是任意函数. 对λ阶分数阶(p,q)-积分定义为

$$\mathrm{I}_{p,q}^{\lambda}w(t) = \frac{1}{p^{\binom{\lambda}{2}}\Gamma_{p,q}(\lambda)}\int_0^t (t-qs)_{p,q}^{(\lambda-1)}w\left(\frac{s}{p^{\lambda-1}}\right)\mathrm{d}_{p,q}s,$$

且$\mathrm{I}_{p,q}^{0}w(t) = w(t)$.

定义 6（Soontharanon et al.，2020） 设$\lambda\in(0,1]$，$0<q<p<1$，且w为定义在$[0,1]$上的任意函数. 对λ阶Caputo型分数阶(p,q)-导数定义为

$${}^cD_{p,q}^{\lambda}w(t) = \mathrm{I}_{p,q}^{1-\lambda}D_{p,q}w(t) = \frac{1}{p^{\binom{1-\lambda}{2}}\Gamma_{p,q}(1-\lambda)}\int_0^t (t-qs)_{p,q}^{(-\lambda)}D_{p,q}w\left(\frac{s}{p^{-\lambda}}\right)\mathrm{d}_{p,q}s,$$

且${}^cD_{p,q}^{0}w(t) = w(t)$.

引理 1（Soontharanon et al.，2020） 对$\lambda\in(m-1,m]$，$m\in\mathbb{N}$，$0<q<p\leqslant 1$且w: $[0,1]\to\mathbb{R}$，有

$$\mathrm{I}_{p,q}^{\lambda}({}^cD_{p,q}^{\lambda}w(t)) = w(t) - \sum_{k=0}^{m-1}\frac{t^k}{p^{\binom{\lambda}{2}}\Gamma_{p,q}(k+1)}D_{p,q}^k w(0).$$

特别地，当方程${}^cD_{p,q}^{\lambda}w(t) = 0$时，其通解可以表示为

$$w(t) = c_0 + c_1 t + c_2 t + \cdots + c_{m-1}t^{m-1},$$

其中$c_0, c_1, c_2, \cdots, c_{m-1}\in\mathbb{R}$. 此外，

$${}^cD_{p,q}^{\lambda}\mathrm{I}_{p,q}^{\lambda}w(t) = w(t).$$

引理 2（Granas et al. ，2003）（Banach不动点定理） 设X是Banach空间E的非空闭凸子集，S: $X\to X$是

压缩算子，则存在唯一的$u^* \in X$，其中$Su^* = u^*$.

引理3(Krasnoselskii，1955)(Krasnoselskii不动点定理)　设Π是Banach空间S的一个非空、凸、闭且有界的子集. 假设算子Ψ_1和Ψ_2将Π映射到S，且满足(i)Ψ_1在Π上是压缩映射;(ii)Ψ_2在Π上是全连续算子;(iii)对任意$s, w \in \Pi$，有$\Psi_1 s + \Psi_2 s \in \Pi$. 则存在$s \in \Pi$使得$s = \Psi_1 s + \Psi_2 s$.

引理4(Katugampola，2014)(Arzela-Ascoli定理)　若E为Banach空间，C是E的紧集，序列$\{x_n\}$在C中一致有界且等度连续，则此序列在C中有一致连续子序列.

引理5　设$0 < q < p \leqslant 1$，则$\boldsymbol{y}(t) \in C([0,1], \mathbb{R}^n)$是系统(1)的解当且仅当$\boldsymbol{y}(t)$满足方程

$$\begin{aligned}\boldsymbol{y}(t) = {}&\boldsymbol{f}_1\big(t, \boldsymbol{y}(t)\big) + \boldsymbol{A}I_{p,q}^{\tau+\alpha+\omega+\theta}\boldsymbol{f}_2\Big(p^{(\tau+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big) + \boldsymbol{B}I_{p,q}^{\delta+\alpha+\omega+\theta}\boldsymbol{f}_3\Big(p^{(\delta+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big)\\ &+ \frac{\boldsymbol{C}}{p^{\binom{\beta+\alpha+\omega+\theta}{2}}\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^{\xi}(\xi - qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\boldsymbol{y}(ps)\,\mathrm{d}_{p,q}s - \boldsymbol{f}_1\big(0, \boldsymbol{C}I_{p,q}^{\beta+\alpha+\omega+\theta}\boldsymbol{y}(\xi)\big).\end{aligned} \tag{2}$$

证明　必要性证明如下：在系统(1)中第一个方程的两侧应用算子$I_{p,q}^{\alpha}$，并取引理1中$m = 1$的情形，存在$c_0 \in \mathbb{R}$，有

$$\begin{aligned}{}^{c}D_{p,q}^{\omega}\Big({}^{c}D_{p,q}^{\theta}\big(y(t) - \boldsymbol{f}_1(t, \boldsymbol{y}(t))\big)\Big) = {}&\boldsymbol{A}I_{p,q}^{\tau+\alpha}\boldsymbol{f}_2\Big(p^{(\tau+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big)\\ &+ \boldsymbol{B}I_{p,q}^{\delta+\alpha}\boldsymbol{f}_3\Big(p^{(\delta+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big) + c_0,\end{aligned}$$

$$\begin{aligned}{}^{c}D_{p,q}^{\theta}\big(\boldsymbol{y}(t) - \boldsymbol{f}_1(t, \boldsymbol{y}(t))\big) = {}&\boldsymbol{A}I_{p,q}^{\tau+\alpha+\omega}\boldsymbol{f}_2\Big(p^{(\tau+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big)\\ &+ \boldsymbol{B}I_{p,q}^{\delta+\alpha+\omega}\boldsymbol{f}_3\Big(p^{(\delta+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big) + I_{p,q}^{\omega}c_0 + c_1,\end{aligned}$$

$$\begin{aligned}\boldsymbol{y}(t) - \boldsymbol{f}_1\big(t, \boldsymbol{y}(t)\big) = {}&\boldsymbol{A}I_{p,q}^{\tau+\alpha+\omega+\theta}\boldsymbol{f}_2\Big(p^{(\tau+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big)\\ &+ \boldsymbol{B}I_{p,q}^{\delta+\alpha+\omega+\theta}\boldsymbol{f}_3\Big(p^{(\delta+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big) + I_{p,q}^{\omega+\theta}c_0 + I_{p,q}^{\theta}c_1 + c_2\\ = {}&\boldsymbol{A}I_{p,q}^{\tau+\alpha+\omega+\theta}\boldsymbol{f}_2\Big(p^{(\tau+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big) + \boldsymbol{B}I_{p,q}^{\delta+\alpha+\omega+\theta}\boldsymbol{f}_3\Big(p^{(\delta+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big)\\ &+ \frac{c_0 t^{\omega+\theta}}{p^{\binom{\omega+\theta}{2}}\Gamma_{p,q}(\omega+\theta+1)} + \frac{c_1 t^{\theta}}{p^{\binom{\theta}{2}}\Gamma_{p,q}(\theta+1)} + c_2,\end{aligned}$$

其中$c_2 = \boldsymbol{y}(0) - \boldsymbol{f}_1\big(0, \boldsymbol{y}(0)\big)$.

根据系统(1)的初值条件$\boldsymbol{y}(0) = \dfrac{\boldsymbol{C}}{p^{\binom{\beta+\alpha+\omega+\theta}{2}}\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\displaystyle\int_0^{\xi}(\xi - qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\boldsymbol{y}(ps)\,\mathrm{d}_{p,q}s$，得

$$c_2 = \frac{\boldsymbol{C}}{p^{\binom{\beta+\alpha+\omega+\theta}{2}}\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^{\xi}(\xi - qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\boldsymbol{y}(ps)\,\mathrm{d}_{p,q}s - \boldsymbol{f}_1\big(0, \boldsymbol{C}I_{p,q}^{\beta+\alpha+\omega+\theta}\boldsymbol{y}(\xi)\big).$$

再根据系统(1)的初值条件${}^{c}D_{p,q}^{\omega}\Big({}^{c}D_{p,q}^{\theta}\big(\boldsymbol{y}(t) - \boldsymbol{f}_1(t, \boldsymbol{y}(t))\big)\Big)\Big|_{t=0} = 0$，得$c_0 = 0$. 最后根据系统(1)的初值条件${}^{c}D_{p,q}^{\theta}\big(\boldsymbol{y}(t) - \boldsymbol{f}_1(t, \boldsymbol{y}(t))\big)\Big|_{t=0} = 0$，得$c_1 = 0$. 因此

$$\begin{aligned}\boldsymbol{y}(t) = {}&\boldsymbol{f}_1\big(t, \boldsymbol{y}(t)\big) + \boldsymbol{A}I_{p,q}^{\tau+\alpha+\omega+\theta}\boldsymbol{f}_2\Big(p^{(\tau+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big) + \boldsymbol{B}I_{p,q}^{\delta+\alpha+\omega+\theta}\boldsymbol{f}_3\Big(p^{(\delta+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big)\\ &+ \frac{\boldsymbol{C}}{p^{\binom{\beta+\alpha+\omega+\theta}{2}}\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^{\xi}(\xi - qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\boldsymbol{y}(ps)\,\mathrm{d}_{p,q}s - \boldsymbol{f}_1\big(0, \boldsymbol{C}I_{p,q}^{\beta+\alpha+\omega+\theta}\boldsymbol{y}(\xi)\big).\end{aligned}$$

另一方面，对于充分性证明，在方程(2)的两边同时应用${}^{c}D_{p,q}^{\theta}$，得

$$\begin{aligned}{}^{c}D_{p,q}^{\theta}\big(\boldsymbol{y}(t) - \boldsymbol{f}_1(t, \boldsymbol{y}(t))\big) = {}&\boldsymbol{A}\,{}^{c}D_{p,q}^{\theta}I_{p,q}^{\tau+\alpha+\omega+\theta}\boldsymbol{f}_2\Big(p^{(\tau+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big)\\ &+ \boldsymbol{B}\,{}^{c}D_{p,q}^{\theta}I_{p,q}^{\delta+\alpha+\omega+\theta}\boldsymbol{f}_3\Big(p^{(\delta+\alpha+\omega+\theta)}t, \boldsymbol{y}\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big)\\ &+ {}^{c}D_{p,q}^{\theta}\frac{\boldsymbol{C}}{p^{\binom{\beta+\alpha+\omega+\theta}{2}}\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^{\xi}(\xi - qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\boldsymbol{y}(ps)\,\mathrm{d}_{p,q}s\\ &- {}^{c}D_{p,q}^{\theta}\boldsymbol{f}_1\big(0, \boldsymbol{C}I_{p,q}^{\beta+\alpha+\omega+\theta}\boldsymbol{y}(\xi)\big).\end{aligned}$$

故

$$ {}^{c}D_{p,q}^{\theta}\Big(y(t)-f_1\big(t,y(t)\big)\Big)=AI_{p,q}^{\tau+\alpha+\omega}f_2\Big(p^{(\tau+\alpha+\omega+\theta)}t,y\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big)+BI_{p,q}^{\delta+\alpha+\omega}f_3\Big(p^{(\delta+\alpha+\omega+\theta)}t,y\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big). \quad (3)$$

在方程(3)的两边同时应用${}^{c}D_{p,q}^{\omega}$，得

$$ {}^{c}D_{p,q}^{\omega}\Big({}^{c}D_{p,q}^{\theta}\Big(y(t)-f_1\big(t,y(t)\big)\Big)\Big)=AI_{p,q}^{\tau+\alpha}f_2\Big(p^{(\tau+\alpha+\omega+\theta)}t,y\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big) + BI_{p,q}^{\delta+\alpha}f_3\Big(p^{(\delta+\alpha+\omega+\theta)}t,y\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big). \quad (4)$$

在方程(4)的两边同时应用${}^{c}D_{p,q}^{\alpha}$，得

$$ {}^{c}D_{p,q}^{\alpha}\Big({}^{c}D_{p,q}^{\omega}\Big({}^{c}D_{p,q}^{\theta}\Big(y(t)-f_1\big(t,y(t)\big)\Big)\Big)\Big)=A\,{}^{c}D_{p,q}^{\alpha}I_{p,q}^{\tau+\alpha}f_2\Big(p^{(\tau+\alpha+\omega+\theta)}t,y\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big) + B\,{}^{c}D_{p,q}^{\alpha}I_{p,q}^{\delta+\alpha}f_3\Big(p^{(\delta+\alpha+\omega+\theta)}t,y\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big).$$

因此

$$ {}^{c}D_{p,q}^{\alpha}\Big({}^{c}D_{p,q}^{\omega}\Big({}^{c}D_{p,q}^{\theta}\Big(y(t)-f_1\big(t,y(t)\big)\Big)\Big)\Big)=AI_{p,q}^{\tau}f_2\Big(p^{(\tau+\alpha+\omega+\theta)}t,y\big(p^{(\tau+\alpha+\omega+\theta)}t\big)\Big)+BI_{p,q}^{\delta}f_3\Big(p^{(\delta+\alpha+\omega+\theta)}t,y\big(p^{(\delta+\alpha+\omega+\theta)}t\big)\Big).$$

显然

$$ y(0)=\frac{C}{p^{\binom{\beta+\alpha+\omega+\theta}{2}}\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^{\xi}(\xi-qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}y(ps)\,\mathrm{d}_{p,q}s,$$

$$ {}^{c}D_{p,q}^{\omega}\Big({}^{c}D_{p,q}^{\theta}\Big(y(t)-f_1\big(t,y(t)\big)\Big)\Big)\Big|_{t=0}=0,\qquad {}^{c}D_{p,q}^{\theta}\Big(y(t)-f_1\big(t,y(t)\big)\Big)\Big|_{t=0}=0.$$

证毕.

2 解的存在唯一性

对于向量$\boldsymbol{y}=(y_1,y_2,\cdots,y_n)\in\mathbb{R}^n$和矩阵$\boldsymbol{A}=(a_{ij})\in\mathbb{R}^{n\times n}$，定义欧几里得向量函数$\|\boldsymbol{y}\|=\sum_{i=1}^{n}|y_i|$和矩阵范数$\|\boldsymbol{A}\|=\max_{1\leqslant j\leqslant n}\sum_{i=1}^{n}|a_{ij}|$. 设$\Xi$为所有从$[0,1]$到$\mathbb{R}^n$的向量值连续函数构成的Banach空间，其范数定义为$\|\boldsymbol{y}\|_{\Xi}=\max_{t\in[0,1]}\|\boldsymbol{y}(t)\|$.

将系统(1)转化为不动点问题$\boldsymbol{y}=\Lambda\boldsymbol{y}$，其中算子$\Lambda:\Xi\to\Xi$定义为

$$ (\Lambda\boldsymbol{y})(t)=\boldsymbol{f}_1\big(t,\boldsymbol{y}(t)\big)+\frac{\boldsymbol{A}}{p^{\binom{\tau+\alpha+\omega+\theta}{2}}\Gamma_{p,q}(\tau+\alpha+\omega+\theta)}\int_0^t(t-qs)_{p,q}^{(\tau+\alpha+\omega+\theta-1)}\boldsymbol{f}_2\big(ps,\boldsymbol{y}(ps)\big)\mathrm{d}_{p,q}s + \frac{\boldsymbol{B}}{p^{\binom{\delta+\alpha+\omega+\theta}{2}}\Gamma_{p,q}(\delta+\alpha+\omega+\theta)}\int_0^t(t-qs)_{p,q}^{(\delta+\alpha+\omega+\theta-1)}\boldsymbol{f}_3\big(ps,\boldsymbol{y}(ps)\big)\mathrm{d}_{p,q}s + \frac{\boldsymbol{C}}{p^{\binom{\beta+\alpha+\omega+\theta}{2}}\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^{\xi}(\xi-qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\boldsymbol{y}(ps)\,\mathrm{d}_{p,q}s-\boldsymbol{f}_1\big(0,\boldsymbol{C}I_{p,q}^{\beta+\alpha+\omega+\theta}\boldsymbol{y}(\xi)\big). \quad (5)$$

故系统(1)有解转化为算子Λ存在不动点.

在证明主要结果之前，给出以下假设

(H_1)对于所有的$\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^n$和$t\in[0,1]$，存在Lipschitz常数$L_1$使得

$$\big\|\boldsymbol{f}_1(t,\boldsymbol{y})-\boldsymbol{f}_1(t,\boldsymbol{z})\big\|_{\Xi}\leqslant L_1\|\boldsymbol{y}-\boldsymbol{z}\|_{\Xi}.$$

(H_2)对于所有的$\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^n$和$t\in[0,1]$，存在Lipschitz常数$L_2$使得

$$\big\|\boldsymbol{f}_2(t,\boldsymbol{y})-\boldsymbol{f}_2(t,\boldsymbol{z})\big\|_{\Xi}\leqslant L_2\|\boldsymbol{y}-\boldsymbol{z}\|_{\Xi}.$$

(H_3)对于所有的$\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^n$和$t\in[0,1]$，存在Lipschitz常数$L_3$使得

$$\big\|\boldsymbol{f}_3(t,\boldsymbol{y})-\boldsymbol{f}_3(t,\boldsymbol{z})\big\|_{\Xi}\leqslant L_3\|\boldsymbol{y}-\boldsymbol{z}\|_{\Xi}.$$

定理1　假设条件$(H_1)\sim(H_3)$成立，且如下条件也成立

$$\Omega \coloneqq L_1+\frac{\|\boldsymbol{A}\|L_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)}+\frac{\|\boldsymbol{B}\|L_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)}$$
$$+\frac{\|\boldsymbol{C}\|\xi^{(\beta+\alpha+\omega+\theta)}(1+L_1)}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta+1)}<1. \tag{6}$$

则系统(1)在区间$[0,1]$上存在唯一的连续解.

证明　为验证Banach压缩映射原理的条件，采用两步验证法.

设$K_j=\max\limits_{t\in[0,1]}\|f_j(t,0)\|<+\infty, j=1,2,3$, 并考虑闭球$B_r=\left\{\boldsymbol{y}\in\Xi:\|\boldsymbol{y}\|_{\Xi}\leqslant r\right\}$, 其中

$$K_1+\frac{\|\boldsymbol{A}\|K_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)}+\frac{\|\boldsymbol{B}\|K_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)}\leqslant(1-\Omega)r.$$

根据条件可知

$$\left\|f_j(t,\boldsymbol{y})\right\|_{\Xi}\leqslant\left\|f_j(t,\boldsymbol{y})-f_j(t,0)\right\|_{\Xi}+\left\|f_j(t,0)\right\|_{\Xi}\leqslant L_jr+K_j,\ j=1,2,3.$$

第一步：证明$\Lambda B_r\subset B_r$. 对$t\in B_r, t\in[0,1]$, 有

$$\begin{aligned}
\left\|(\Lambda\boldsymbol{y})(t)\right\|\leqslant&\left\|f_1(t,\boldsymbol{y}(t))\right\|+\frac{\|\boldsymbol{A}\|}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta)}\int_0^t(t-qs)_{p,q}^{(\tau+\alpha+\omega+\theta-1)}\left\|f_2(ps,\boldsymbol{y}(ps))\right\|\mathrm{d}_{p,q}s\\
&+\frac{\|\boldsymbol{B}\|}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta)}\int_0^t(t-qs)_{p,q}^{(\delta+\alpha+\omega+\theta-1)}\left\|f_3(ps,\boldsymbol{y}(ps))\right\|\mathrm{d}_{p,q}s\\
&+\frac{\|\boldsymbol{C}\|}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^\xi(\xi-qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\left\|\boldsymbol{y}(ps)\right\|\mathrm{d}_{p,q}s+\left\|f_1\left(0,CI_{p,q}^{\beta+\alpha+\omega+\theta}\boldsymbol{y}(\xi)\right)\right\|\\
\leqslant&L_1r+K_1+\frac{\|A\|(L_2r+K_2)}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)}+\frac{\|\boldsymbol{B}\|(L_3r+K_3)}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)}\\
&+\frac{\|\boldsymbol{C}\|\xi^{(\beta+\alpha+\omega+\theta)}(1+L_1)r}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta+1)}+K\\
=&\Omega r+K_1+\frac{\|\boldsymbol{A}\|K_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)}+\frac{\|\boldsymbol{B}\|K_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)}\\
\leqslant&\Omega r+(1-\Omega)r=r.
\end{aligned}$$

因此，$\|\Lambda y\|_{\Xi}\leqslant r$. 故$\Lambda B_r\subset B_r$.

第二步：对任意$\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^n$, $t\in[0,1]$,有

$$\begin{aligned}
&\left\|(\Lambda\boldsymbol{y})(t)-(\Lambda\boldsymbol{z})(t)\right\|\\
\leqslant&\left\|f_1(t,\boldsymbol{y}(t))-f_1(t,\boldsymbol{z}(t))\right\|\\
&+\frac{\|\boldsymbol{A}\|}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta)}\int_0^t(t-qs)_{p,q}^{(\tau+\alpha+\omega+\theta-1)}\left\|f_2(ps,\boldsymbol{y}(ps))-f_2(ps,\boldsymbol{z}(ps))\right\|\mathrm{d}_{p,q}s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\|B\|}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta)}\int_0^t (t-qs)_{p,q}^{(\delta+\alpha+\omega+\theta-1)}\left\| f_3\big(ps,y(ps)\big)-f_3\big(ps,z(ps)\big)\right\| \mathrm{d}_{p,q}s\\
&+\frac{\|C\|}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^\xi (\xi-qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\left\| y(ps)-z(ps)\right\| \mathrm{d}_{p,q}s\\
&+\left\| f_1\big(0, CI_{p,q}^{\beta+\alpha+\omega+\theta}y(\xi)\big)-f_1\big(0, CI_{p,q}^{\beta+\alpha+\omega+\theta}z(\xi)\big)\right\|\\
\leqslant\, & L_1\|y-z\|_\Xi+\frac{\|A\|L_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)}\|y-z\|_\Xi+\frac{\|B\|L_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)}\|y-z\|_\Xi\\
&+\frac{\|C\|\xi^{\beta+\alpha+\omega+\theta}(1+L_1)}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta+1)}\|y-z\|_\Xi.
\end{aligned}
$$

因此，

$$\|\Lambda y-\Lambda z\|_\Xi\leqslant\Omega\|y-z\|_\Xi.$$

由于$\Omega<1$，故由Banach压缩映射原理可知Λ是压缩的，并且有唯一不动点，即系统(1)有唯一解. 证毕.

定理2 假设条件(H_1)成立，且如下条件也成立

$$\Delta := L_1+\frac{\|C\|\xi^{(\beta+\alpha+\omega+\theta)}(1+L_1)}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta+1)}<1, \tag{7}$$

则系统(1)至少存在一个解.

证明 对任意常数R，考虑集合$\Pi=\left\{y\in\Xi:\|y\|_\Xi\leqslant R\right\}$，其中

$$
\begin{aligned}
R\geqslant&\left(K_1+\frac{\|A\|\tilde{L}_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)}+\frac{\|B\|\tilde{L}_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)}\right)\\
&\times\left(1-L_1-\frac{\|C\|(1+L_1)\xi^{(\beta+\alpha+\omega+\theta)}}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta+1)}\right)^{-1},
\end{aligned}
$$

则B_R是Banach空间Ξ上的有界闭凸非空子集.

定义B_R上的两个算子Ψ_1,Ψ_2为

$$
\begin{aligned}
(\Psi_1y)(t)=&f_1\big(t,y(t)\big)+\frac{C}{p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^\xi(\xi-qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}y(ps)\mathrm{d}_{p,q}s\\
&-f_1\big(0,CI_{p,q}^{\beta+\alpha+\omega+\theta}y(\xi)\big),\\
(\Psi_2y)(t)=&\frac{A}{p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\Gamma_{p,q}(\tau+\alpha+\omega+\theta)}\int_0^t(t-qs)_{p,q}^{(\tau+\alpha+\omega+\theta-1)}f_2\big(ps,y(ps)\big)\mathrm{d}_{p,q}s\\
&+\frac{B}{p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\Gamma_{p,q}(\delta+\alpha+\omega+\theta)}\int_0^t(t-qs)_{p,q}^{(\delta+\alpha+\omega+\theta-1)}f_3\big(ps,y(ps)\big)\mathrm{d}_{p,q}s.
\end{aligned}
$$

若对所有$(t,y)\in[0,1]\times\Pi$，$f_2(t,y)$，$f_3(t,y)$是有界的，对任意$y\in\Pi$和$t\in[0,1]$，表示

$$\tilde{L}_2=\max_{(t,y)\in[0,1]\times\Pi}\left\|f_2(t,y)\right\|,\qquad \tilde{L}_3=\max_{(t,y)\in[0,1]\times\Pi}\left\|f_3(t,y)\right\|.$$

第一步：对任意$y,z\in\Pi$且$t\in[0,1]$，有

$$\left\|\Psi_1y+\Psi_2z\right\|_{\Xi}\leqslant L_1R+\frac{\|C\|\xi^{(\beta+\alpha+\omega+\theta)}(1+L_1)R}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta+1)}+K_1+\frac{\|A\|\tilde{L}_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)}$$
$$+\frac{\|B\|\tilde{L}_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)}\leqslant R.$$

因此，$\Psi_1y+\Psi_2z\in\Pi$，f_2，f_3的连续性意味着算子Ψ_2是连续的．

此外，由于

$$\left\|\Psi_2z\right\|_{\Xi}\leqslant\frac{\|A\|\tilde{L}_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)}+\frac{\|B\|\tilde{L}_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)},$$

则Ψ_2在Π上一致有界．

第二步：对任意$z\in\Pi$且$t_1,t_2\in[0,1]$，有

$$\begin{aligned}&\left\|(\Psi_2z)(t_2)-(\Psi_2z)(t_1)\right\|\\
\leqslant&\frac{\|A\|\tilde{L}_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta)}\left|\int_0^{t_2}(t_2-qs)_{p,q}^{(\tau+\alpha+\omega+\theta-1)}\mathrm{d}_{p,q}s-\int_0^{t_1}(t_1-qs)_{p,q}^{(\tau+\alpha+\omega+\theta-1)}\mathrm{d}_{p,q}s\right|\\
&+\frac{\|B\|\tilde{L}_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta)}\left|\int_0^{t_2}(t_2-qs)_{p,q}^{(\delta+\alpha+\omega+\theta-1)}\mathrm{d}_{p,q}s-\int_0^{t_1}(t_1-qs)_{p,q}^{(\delta+\alpha+\omega+\theta-1)}\mathrm{d}_{p,q}s\right|\\
\leqslant&\frac{\|A\|\tilde{L}_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta)}\int_0^{t_1}\left|(t_2-qs)_{p,q}^{(\tau+\alpha+\omega+\theta-1)}-(t_1-qs)_{p,q}^{(\tau+\alpha+\omega+\theta-1)}\right|\mathrm{d}_{p,q}s\\
&+\frac{\|A\|\tilde{L}_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta)}\int_{t_1}^{t_2}(t_2-qs)_{p,q}^{(\tau+\alpha+\omega+\theta-1)}\mathrm{d}_{p,q}s\\
&+\frac{\|B\|\tilde{L}_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta)}\int_0^{t_1}\left|(t_2-qs)_{p,q}^{(\delta+\alpha+\omega+\theta-1)}-(t_1-qs)_{p,q}^{(\delta+\alpha+\omega+\theta-1)}\right|\mathrm{d}_{p,q}s\\
&+\frac{\|B\|\tilde{L}_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta)}\int_{t_1}^{t_2}(t_2-qs)_{p,q}^{(\delta+\alpha+\omega+\theta-1)}\mathrm{d}_{p,q}s\,.\end{aligned}$$

这意味着表达式与z无关，且当$t_2\to t_1$时，有$\left\|(\Psi_2z)(t_2)-(\Psi_2z)(t_1)\right\|\to0$.

故Ψ_2在Π上是相对紧的．因此由于Arzelà–Ascoli定理可知，Ψ_2在Π上是紧的．

第三步：对任意$y,z\in\Pi$且$t\in[0,1]$，有

$$\left\|\Psi_1y-\Psi_1z\right\|_{\Xi}\leqslant\left(L_1+\frac{\|C\|\xi^{(\beta+\alpha+\omega+\theta)}(1+L_1)}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta+1)}\right)\left\|y-z\right\|_{\Xi}.$$

根据条件(7)可以得出Ψ_1是压缩的．

因此，根据引理3可以得出系统(1)在区间$[0,1]$上至少存在一个连续解．证毕．

3 解的有限时间稳定性

在本节中,介绍系统(1)的解的稳定性条件并验证解的有限时间稳定性.

定义 7(Dorato, 2006) 系统(1)关于非局部值是有限时间稳定的,如果对系统(1)的两个解$\boldsymbol{x},\boldsymbol{y}$,存在$\mu>0$,使得

$$\left\|\boldsymbol{x}(t)-\boldsymbol{y}(t)\right\|\leqslant\mu\left\|\boldsymbol{x}(p\zeta)-\boldsymbol{y}(p\zeta)\right\|,$$

其中$0<\zeta<1$,与非局部条件有关.

定理 3 假设条件(H_1)~(H_3)成立,且条件(6)成立.则系统(1)的解在非局部初值下具有有限时间稳定性.

证明 考虑系统(1)的两个解$\boldsymbol{y}_1,\boldsymbol{y}_2$,并满足不动点方程$\Lambda\boldsymbol{y}=\boldsymbol{y}$,其中$\Lambda$由式(2)定义,利用定理3中的条件得

$$\begin{aligned}
\left\|\boldsymbol{y}_1(t)-\boldsymbol{y}_2(t)\right\|\leqslant&\left\|\boldsymbol{f}_1(t,\boldsymbol{y}_1(t))-\boldsymbol{f}_1(t,\boldsymbol{y}_2(t))\right\|\\
&+\frac{\|\boldsymbol{A}\|}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta)}\int_0^t(t-qs)_{p,q}^{(\tau+\alpha+\omega+\theta-1)}\left\|\boldsymbol{f}_2(ps,\boldsymbol{y}_1(ps))-\boldsymbol{f}_2(ps,\boldsymbol{y}_2(ps))\right\|\mathrm{d}_{p,q}s\\
&+\frac{\|\boldsymbol{B}\|}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta)}\int_0^t(t-qs)_{p,q}^{(\delta+\alpha+\omega+\theta-1)}\left\|\boldsymbol{f}_3(ps,\boldsymbol{y}_1(ps))-\boldsymbol{f}_3(ps,\boldsymbol{y}_2(ps))\right\|\mathrm{d}_{p,q}s\\
&+\left\|\boldsymbol{y}_1(0)-\boldsymbol{y}_2(0)\right\|+\left\|\boldsymbol{f}_1(0,\boldsymbol{y}_1(0))-\boldsymbol{f}_1(0,\boldsymbol{y}_2(0))\right\|\\
\leqslant&L_1\left\|\boldsymbol{y}_1(t)-\boldsymbol{y}_2(t)\right\|+\frac{\|\boldsymbol{A}\|L_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)}\left\|\boldsymbol{y}_1(t)-\boldsymbol{y}_2(t)\right\|\\
&+\frac{\|\boldsymbol{B}\|L_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)}\left\|\boldsymbol{y}_1(t)-\boldsymbol{y}_2(t)\right\|+(1+L_2)\left\|\boldsymbol{y}_1(0)-\boldsymbol{y}_2(0)\right\|\\
=&(L_1+E_1+E_2)\left\|\boldsymbol{y}_1(t)-\boldsymbol{y}_2(t)\right\|+(1+L_2)\left\|\boldsymbol{y}_1(0)-\boldsymbol{y}_2(0)\right\|.
\end{aligned}$$

其中

$$E_1=\frac{\|\boldsymbol{A}\|L_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)},\qquad E_2=\frac{\|\boldsymbol{B}\|L_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)}.$$

此外

$$\begin{aligned}
\left\|\boldsymbol{y}_1(0)-\boldsymbol{y}_2(0)\right\|\leqslant&\frac{\|\boldsymbol{C}\|}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^\xi(\xi-qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\left\|\boldsymbol{y}_1(ps)-\boldsymbol{y}_2(ps)\right\|\mathrm{d}_{p,q}s\\
\leqslant&\frac{\|\boldsymbol{C}\|}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta)}\int_0^\xi(\xi-qs)_{p,q}^{(\beta+\alpha+\omega+\theta-1)}\mathrm{d}_{p,q}s\left\|\boldsymbol{y}_1(p\xi)-\boldsymbol{y}_2(p\xi)\right\|\\
=&\frac{\|\boldsymbol{C}\|\xi^{\beta+\alpha+\omega+\theta}}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta+1)}\left\|\boldsymbol{y}_1(p\xi)-\boldsymbol{y}_2(p\xi)\right\|.
\end{aligned}$$

故

$$\left\|\boldsymbol{y}_1(0)-\boldsymbol{y}_2(0)\right\|\leqslant E_3\left\|\boldsymbol{y}_1(p\xi)-\boldsymbol{y}_2(p\xi)\right\|,$$

其中$E_3=\dfrac{\|\boldsymbol{C}\|\xi^{\beta+\alpha+\omega+\theta}}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\beta+\alpha+\omega+\theta+1)}$.

因此,综合所有项,满足

$$\left\|\boldsymbol{y}_1(t)-\boldsymbol{y}_2(t)\right\| \leqslant \frac{(1+L_2)E_3}{1-(L_1+E_1+E_2)}\left\|\boldsymbol{y}_1(p\xi)-\boldsymbol{y}_2(p\xi)\right\|.$$

由于条件(6)成立,则$L_1+E_1+E_2<1$,故系数$\mu \coloneqq \dfrac{(1+L_2)E_3}{1-(L_1+E_1+E_2)}$有界且$\mu>0$,则系统(1)的解在非局部初值下具有有限时间稳定性.

4　例　子

例 1　考虑分数阶混合(p,q)-积分差分系统(1),其中$\boldsymbol{y}=(y_1,y_2)^{\mathrm{T}}$, $\alpha=\dfrac{1}{3}$, $\omega=\dfrac{1}{32}$, $\theta=\dfrac{3}{32}$, $\tau=\dfrac{5}{16}$, $\delta=\dfrac{5}{16}$, $\beta=\dfrac{1}{16}$, $\xi=\dfrac{3}{4}$, $q=\{0.15,0.1,0.18\}$, $p=\{0.25,0.3,0.47\}$, 且

$$\boldsymbol{A}=\begin{pmatrix}\frac{1}{14} & 0\\ \frac{1}{14} & \frac{1}{7}\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}0 & \frac{1}{12}\\ \frac{1}{6} & \frac{1}{12}\end{pmatrix},\quad \boldsymbol{C}=\begin{pmatrix}0 & 0\\ \frac{1}{9} & 0\end{pmatrix}.$$

对于函数,令

$$f_1(t,\boldsymbol{y})=\frac{t}{20}(\cos y_1,\cos y_2)^{\mathrm{T}},\quad f_2(t,\boldsymbol{y})=\frac{t^2}{7}(\sin y_1,\sin y_2)^{\mathrm{T}},\quad f_3(t,\boldsymbol{y})=\left(\frac{1}{9}\frac{|y_1|}{|y_1|+1},\tan t\right)^{\mathrm{T}}.$$

经计算$\|\boldsymbol{A}\|=\dfrac{1}{7}$, $\|\boldsymbol{B}\|=\dfrac{1}{6}$, $\|\boldsymbol{C}\|=\dfrac{1}{9}$, $L_1=\dfrac{1}{20}$, $L_2=\dfrac{1}{7}$, $L_3=\dfrac{1}{9}$.

由表1~3中$\omega_1,\omega_2,\omega_3$中的数值结果可以得出

$$\omega_1=\frac{\|\boldsymbol{A}\|L_2}{\left|p^{\left(\frac{\tau+\alpha+\omega+\theta}{2}\right)}\right|\Gamma_{p,q}(\tau+\alpha+\omega+\theta+1)}\approx\begin{cases}0.039\ 2, & p=0.25, q=0.15,\\ 0.066\ 8, & p=0.3, q=0.1,\\ 0.089\ 0, & p=0.47, q=0.18,\end{cases}$$

表 1　$\omega_1,\omega_2,\omega_3,\Omega,\Delta$在$p=0.25$, $q=0.15$时的数值结果[1)]

Table 1　Numerical results of $\omega_1,\omega_2,\omega_3,\Omega,\Delta$ for $p=0.25$, $q=0.15$

n	Ω	Δ	ω_1	ω_2	ω_3	n	Ω	Δ	ω_1	ω_2	ω_3
1	0.542 0	0.425 9	0.060 9	0.055 2	0.375 9	14	0.367 4	0.292 7	0.039 2	0.035 6	0.242 7
2	0.454 3	0.359 0	0.049 9	0.045 3	0.309 0	15	0.367 4	0.292 7	0.039 2	$\underline{0.035\ 5}$	0.242 7
3	0.414 5	0.328 7	0.045 0	0.040 8	0.278 7	16	0.367 4	$\underline{0.292\ 6}$	0.039 2	0.035 5	$\underline{0.242\ 6}$
4	0.394 0	0.313 0	0.042 5	0.038 5	0.263 0	17	$\underline{0.367\ 3}$	0.292 6	0.039 2	0.035 5	0.242 6
5	0.382 8	0.304 5	0.041 1	0.037 3	0.254 5	18	0.367 3	0.292 6	0.039 2	0.035 5	0.242 6
6	0.376 4	0.299 6	0.040 3	0.036 6	0.249 6	19	0.367 3	0.292 6	0.039 2	0.035 5	0.242 6
7	0.372 7	0.296 7	0.039 8	0.036 1	0.246 7	20	0.367 3	0.292 6	0.039 2	0.035 5	0.242 6
8	0.370 5	0.295 1	0.039 6	0.035 9	0.245 1	21	0.367 3	0.292 6	0.039 2	0.035 5	0.242 6
9	0.369 2	0.294 1	0.039 4	0.035 8	0.244 1	22	0.367 3	0.292 6	0.039 2	0.035 5	0.242 6
10	0.368 5	0.293 5	0.039 3	0.035 7	0.243 5	23	0.367 3	0.292 6	0.039 2	0.035 5	0.242 6
11	0.368 0	0.293 1	0.039 3	0.035 6	0.243 1	24	0.367 3	0.292 6	0.039 2	0.035 5	0.242 6
12	0.367 7	0.292 9	$\underline{0.039\ 2}$	0.035 6	0.242 9	25	0.367 3	0.292 6	0.039 2	0.035 5	0.242 6
13	0.367 5	0.292 8	0.039 2	0.035 6	0.242 8	⋮	⋮	⋮	⋮	⋮	⋮

1) 下划线数字表示后续的值与之相同.

表2 $\omega_1, \omega_2, \omega_3, \Omega, \Delta$ 在 $p = 0.3, q = 0.1$ 时的数值结果[1)]

Table 2 Numerical results of $\omega_1, \omega_2, \omega_3, \Omega, \Delta$ for $p = 0.3, q = 0.1$

n	Ω	Δ	ω_1	ω_2	ω_3	n	Ω	Δ	ω_1	ω_2	ω_3
1	0. 583 6	0. 440 4	0. 075 1	0. 068 1	0. 390 4	14	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
2	0. 543 7	0. 411 4	0. 069 4	0. 063 0	0. 361 4	15	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
3	0. 531 5	0. 402 4	0. 067 7	0. 061 4	0. 352 4	16	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
4	0. 527 5	0. 399 5	0. 067 1	0. 060 9	0. 349 5	17	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
5	0. 526 2	0. 398 6	0. 066 9	0. 060 7	0. 348 6	18	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
6	0. 525 8	0. 398 2	0. 066 9	0. 060 7	0. 348 2	19	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
7	0. 525 6	0. 398 1	0. 066 8	0. 060 6	0. 348 1	20	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
8	0. 525 6	0. 398 1	0. 066 8	0. 060 6	0. 348 1	21	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
9	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1	22	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
10	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1	23	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
11	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1	24	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
12	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1	25	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1
13	0. 525 5	0. 398 1	0. 066 8	0. 060 6	0. 348 1	⋮	⋮	⋮	⋮	⋮	⋮

1)下划线数字表示后续的值与之相同.

表3 $\omega_1, \omega_2, \omega_3, \Omega, \Delta$ 在 $p = 0.47, q = 0.18$ 时的数值结果[1)]

Table 3 Numerical results of $\omega_1, \omega_2, \omega_3, \Omega, \Delta$ for $p = 0.47, q = 0.18$

n	Ω	Δ	ω_1	ω_2	ω_3	n	Ω	Δ	ω_1	ω_2	ω_3
1	0. 735 6	0. 538 4	0. 103 4	0. 093 8	0. 488 4	14	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
2	0. 676 0	0. 496 5	0. 094 1	0. 085 4	0. 446 5	15	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
3	0. 654 9	0. 481 5	0. 090 9	0. 082 5	0. 431 5	16	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
4	0. 647 0	0. 475 9	0. 089 7	0. 081 4	0. 425 9	17	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
5	0. 644 0	0. 473 7	0. 089 3	0. 081 0	0. 423 7	18	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
6	0. 642 8	0. 472 9	0. 089 1	0. 080 8	0. 422 9	19	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
7	0. 642 4	0. 472 6	0. 089 0	0. 080 8	0. 422 6	20	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
8	0. 642 2	0. 472 5	0. 089 0	0. 080 8	0. 422 5	21	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
9	0. 642 2	0. 472 4	0. 089 0	0. 080 8	0. 422 4	22	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
10	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4	23	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
11	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4	24	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
12	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4	25	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4
13	0. 642 1	0. 472 4	0. 089 0	0. 080 7	0. 422 4	⋮	⋮	⋮	⋮	⋮	⋮

1)下划线数字表示后续的值与之相同.

$$\omega_2 = \frac{\|\boldsymbol{B}\| L_3}{\left|p^{\left(\frac{\delta+\alpha+\omega+\theta}{2}\right)}\right| \Gamma_{p,q}(\delta+\alpha+\omega+\theta+1)} \approx \begin{cases} 0.035\,5, & p = 0.25, q = 0.15, \\ 0.060\,6, & p = 0.3, q = 0.1, \\ 0.080\,7, & p = 0.47, q = 0.18, \end{cases}$$

$$\omega_3 = \frac{\|\boldsymbol{C}\| \xi^{(\beta+\alpha+\omega+\theta)}(1+L_1)}{\left|p^{\left(\frac{\beta+\alpha+\omega+\theta}{2}\right)}\right| \Gamma_{p,q}(\beta+\alpha+\omega+\theta+1)} \approx \begin{cases} 0.242\,6, & p = 0.25, q = 0.15, \\ 0.348\,1, & p = 0.3, q = 0.1, \\ 0.422\,4, & p = 0.47, q = 0.18\,. \end{cases}$$

由表1~3中的Ω的数值结果可知

$$\Omega := L_1 + \omega_1 + \omega_2 + \omega_3 \approx \begin{cases} 0.367\,3, & p = 0.25, q = 0.15, \\ 0.525\,5, & p = 0.3, q = 0.1, \\ 0.642\,1, & p = 0.47, q = 0.18. \end{cases}$$

故$\Omega < 1$,因此满足定理1和定理3的所有条件,故所考虑的系统在$[0, 1]$上有唯一解,且该系统的解是有限时间稳定的.

由表1~3中Δ的数值结果可知

$$\Delta := L_1 + \omega_3 \approx \begin{cases} 0.292\,6, & p = 0.25, q = 0.15, \\ 0.398\,1, & p = 0.3, q = 0.1, \\ 0.472\,4, & p = 0.47, q = 0.18. \end{cases}$$

故$\Delta < 1$, 因此满足定理2的所有条件, 故所考虑的系统在$[0, 1]$上至少存在一个解.

5　结　论

本文研究了一类分数阶混合(p,q)-积分差分系统解的存在唯一性以及有限时间稳定性. 在定理1中,利用Banach压缩映射原理得到了给定问题存在唯一解的充分条件;在定理2中,利用Krasnoselskii不动点定理得到了给定问题至少存在一个解的充分条件;在定理3中,验证了该系统解在特定条件下的有限时间稳定性;在最后一节中,给出了一个实例并通过其数值结果验证了主要结论的可行性与适用性. 该分数阶混合(p,q)-积分差分系统的研究成果在多个前沿领域展现出重要的应用价值. 在量子计算领域,系统解的有限时间稳定性特性为设计高效量子控制协议提供了理论基础,能够显著提升量子比特的稳定效率,同时(p,q)-差分算子对离散–连续混合特性的精确描述为非马尔可夫噪声抑制提供了新的数学工具. 在智能控制工程方面,该模型能准确刻画形状记忆合金等智能材料的温度-应变迟滞效应,其稳定性条件确保了材料形变的精确可控性,也为需要同时考虑离散控制信号和连续动力学的机器人运动规划问题提供了解决方案. 在生物医学工程领域,基于(p,q)-差分建立的神经电信号模型可精确描述突触传递的脉冲-电位耦合特性,为癫痫等神经性疾病的信号特征分析提供了新方法,而分数阶积分与差分的结合则为智能给药系统的优化设计提供了量化依据,实现了对药物代谢累积效应和离散给药间隔的协同建模. 这些应用充分体现了该理论模型在处理具有记忆特性和离散-连续耦合特征的复杂系统时的独特优势,为相关领域的工程技术发展提供了重要的理论支撑.

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